Everything is a Harmonic Oscillator

A Gentle Introduction to Quantum Optics

PHY 304 – Advanced Electromagnetism

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1 Introduction

Maxwell's equations and the Lorentz force law tell us everything that we need to know about classical electrodynamics. Specifically, you would have seen how the coupling between the electric and magnetic fields give rise to the dynamic phenomenon of electromagnetic (EM) waves. These waves describe what we experience as light, alongside other useful familiar waves (non-visible portion of the electromagnetic spectrum). Being a 21st century physics student, you would have undoubtedly heard of the existence of photons and the role that they play in being the fundamental particles of light. Despite how natural it feels to describe a light wave as being comprised of photons, the very idea of a photon never appeared in any of our work up to this point, and it seems that there is also no room for it to do so. Specifically, a photon is a discrete object, while the EM waves that we have described thusfar are continuous phenomena permeating the medium in which they lie. I thus ask you the following – where is the photon?

It turns out that the answer to this question involves a great deal of mathematical work, but decades of passing this knowledge from pioneers to students has made it far more pedagogically accessible. This short review is meant to provide that very exposition for the interested reader. The general idea of the process at hand is to ask the natural question[1] of how one can **quantise** the electromagnetic field. This is akin to the quantisation procedure that one may be familiar with from an introductory quantum mechanics course. Do note, however, that this text assumes the reader has some decent knowledge on the contents of electrodynamics in the language of special relativity, specifically the content of Chapter 12 in Griffiths [1].

I also assume the reader has some basic understanding of quantum mechanics, specifically up to the quantum harmonic oscillator. We will use this for the quantisation procedure and later explore some phenomena in quantum optics which cannot be reconciled with classical behaviours, showing some of the neat features of a quantised theory of light.

[1] I say natural, but this is probably natural to literally no one but a physicist.

1.1 The Potential Formalism

As a brief review, let's revisit the potential formalism of electrodynamics as presented by Griffiths (Chapter 10) [1]. We begin with the Maxwell-Gauss law for magnetism:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \implies \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{1.1}$$

where we know that the divergence of a curl identically vanishes, so we can say that $\mathbf{B}$ can be mathematically expressed as the curl of some function, which is the magnetic vector potential. Substituting this into Faraday's law gives:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{A}) \\ &\implies \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \end{aligned} \tag{1.2}$$

where we commute the spatial and temporal derivatives using Schwarz' theorem. Now, we use the fact that the curl of a gradient vanishes identically, to demand that the argument of the curl in Eq. 1.2 can be expressed as the gradient of some scalar function. We thus obtain the electric and magnetic fields in terms of two potential functions:

$$\mathbf{E}(\mathbf{r}, t) = -\boldsymbol{\nabla} V(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) \tag{1.3}$$

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t) \tag{1.4}$$

Since we used two out of four of Maxwell's equations to arrive at the potential expressions, we can now plug these into the remaining two equations to recast electrodynamics in terms of the potentials. We write:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = -\frac{\rho}{\epsilon_0} \implies \nabla^2 V + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{1.5}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \implies \left(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} \right) = -\mu_0 \mathbf{J} \tag{1.6}$$

While these equations look like an absolute nightmare to deal with, they have the exact same informational content as that of Maxwell's four equations. While not mathematically simpler, these equations are more information-dense than the original four equations. Maxwell's four equations required us to find six functions $\{E_x, E_y, E_z, B_x, B_y, B_z\}$ (the fields) when given four functions $\{\rho, J_x, J_y, J_z\}$ (the source distribution). The reduced equations of the potential formalism now only require us to find four functions $\{V, A_x, A_y, A_z\}$ which equivalently produce a unique field solution given the same source distribution. We have thus rid ourselves of two degrees of freedom by exploiting the redundancies encoded into vector calculus/the real world[2]! At this point, it is natural to ask if we can do better than reducing the number of degrees of freedom from six to four, and indeed it's not hard to see that we can definitely do better.

While the electric and magnetic fields are real, observable fields, the underlying potentials are not. It turns out that there is another manifest redundancy in the construction of these potentials too. Specifically, the construction of the vector potential in Eq. 1.1 implies that the resulting $\mathbf{B}$ will not be changed if $\mathbf{A}$ is changed by the gradient of some scalar function. Since this affects the electric field, we also add an unknown scalar function to the electric potential to write:

[2] Depending on whether you ask a mathematician or a physicist.

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda \tag{1.7}$$

$$V' = V + \beta \tag{1.8}$$

The resulting effect on the fields can then be written as:

$$\mathbf{B}' = \boldsymbol{\nabla} \times (\mathbf{A} + \boldsymbol{\nabla}\lambda) = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B} \tag{1.9}$$

$$\mathbf{E}' = -\boldsymbol{\nabla}V - \boldsymbol{\nabla}\beta - \frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\frac{\partial \lambda}{\partial t} \tag{1.10}$$

Demanding that the electric field after the transformation is identical to that from before, we arrive at the condition:

$$\boldsymbol{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0 \Longrightarrow \beta = -\frac{\partial \lambda}{\partial t} \tag{1.11}$$

where the generalisation to the argument of the gradient is made by demanding that this condition should hold at all positions, and thus the entire argument should vanish identically. In Eq. 1.11, we expect a constant of integration as well, but this can be arbitrarily absorbed into the definition of the scalar function λ. Thus, we see that both potentials can be arbitrarily modified up to the derivatives of a scalar function whilst leaving the physical EM fields unchanged. Using the 4-potential $A^\mu = (\rho/c, \mathbf{A})$, we can succinctly write these conditions as:

$$A^\mu \longrightarrow A^{\mu'} = A^\mu + \partial^\mu \lambda \tag{1.12}$$

where $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla})$ is the 4-gradient, sometimes also represented by $\Box$[3]. Squaring this operator produces the (3+1)-dimensional equivalent of the Laplacian, known as the d'Alembertian, which is written as $\Box^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$.[4] This transformation is known as a **gauge transformation**, and this last degree of freedom is consequently known as **gauge freedom**. Evidently, we can rid ourselves of one further degree of freedom before we delve into the mathematical machinery of quantisation.

1.2 Gauge Fixing

The existence of gauge freedom implies that the scalar and vector potentials are defined up to the existence of some scalar function $\lambda(\mathbf{r}, t)$. With this in mind, we can impose a constraint on the scalar and vector potentials *as long as* this imposition admits a unique solution for $\lambda(\mathbf{r}, t)$[5]. For this, there are several useful gauges.

1.2.1 The Coulomb Gauge

In this gauge, we demand that the vector potential is divergenceless:

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{1.13}$$

[3]One should be careful to distinguish the contravariant ∂^μ and covariant ∂_μ forms of the 4-gradient, both of which differ by the sign of the spatial portion since the tensor index is raised/lowered by the metric tensor. This definition also assumes the $(+,-,-,-)$ metric signature, as is standard in quantum field theory and high-energy physics, unlike the $(-,+,+,+)$ signature used in general relativity.

[4]In most quantum field theory textbooks, $\Box$ is used to represent the d'Alembertian itself while ∂^μ is always used to emphasise the 4-vectorial nature of the differential operator.

[5]This prevents us from imposing arbitrarily stupid gauge conditions.

This implies that for a gauge-transformed vector potential (according to Eq. 1.7) to obey this condition, we must have the unknown function λ satisfying:

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{A}' &= \boldsymbol{\nabla}\cdot\mathbf{A} + \nabla^2\lambda = 0 \\ &\implies \lambda(\mathbf{r},t) = \frac{1}{4\pi}\int \mathrm{d}\tau' \, \frac{\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \end{aligned} \tag{1.14}$$

where the solution for $\lambda(\mathbf{r},t)$ is evident since $\boldsymbol{\nabla}\cdot\mathbf{A}$ simply forms a scalar inhomogeneity for the partial differential equation of λ. This produces Poisson's equation for λ with $\boldsymbol{\nabla}\cdot\mathbf{A}$ as the source term, for which the solution is immediate since we know that the Green's function of the Laplacian is simply $1/4\pi|\mathbf{r}-\mathbf{r}'|$. Thus, the existence and uniqueness of $\lambda(\mathbf{r},t)$ is guaranteed, and the gauge is considered viable. In this gauge, we then recover the potentials as:

$$\nabla^2 V(\mathbf{r},t) = -\frac{\rho(\mathbf{r},t)}{\epsilon_0} \implies V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}\tau' \, \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \tag{1.15}$$

$$\Box^2\mathbf{A}(\mathbf{r},t) - \boldsymbol{\nabla}\left(\frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}(\mathbf{r},t) \implies \mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi}\int \mathrm{d}\tau' \, \frac{\delta\left(t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\left[\boldsymbol{\nabla}\left(\frac{1}{c^2}\frac{\partial V(\mathbf{r}',t)}{\partial t}\right) - \mu_0\mathbf{J}(\mathbf{r}',t)\right] \tag{1.16}$$

where I have directly used the fact that the Green's function of the d'Alembertian is $\delta\left(t-\frac{r}{c}\right)/4\pi r$ [2]. While the scalar potential is obtained via a relatively simple (or at least familiar) integral, the vector potential is extremely difficult to solve for in the Coulomb gauge when a source distribution is present[6]. The Coulomb gauge, however, is ideal for semiclassical situations (such as most of atomic physics) where the vector potential $\mathbf{A}$ is to be quantised but the Coulomb interaction is left classical (such as in the Bohr-Sommerfeld model of the atom).

1.2.2 The Lorenz Gauge

In this gauge, we seek some level of symmetry between the scalar and vector potentials' solutions. For this, we impose:

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \longleftrightarrow \partial_\mu A^\mu = 0 \tag{1.17}$$

Then, performing the gauge transformation from Eqs. 1.7 and 1.8, we see that the gauge-transformed potentials must satisfy:

$$\boldsymbol{\nabla}\cdot\mathbf{A}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} = \boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla^2\lambda - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2} = 0 \tag{1.18}$$

On the right hand side (RHS), the first two terms form an inhomogeneity while the last two terms result in a partial differential equation for λ. Specifically, we obtain:

$$\Box\lambda = -\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) \tag{1.19}$$

This is nothing more than an inhomogeneous wave equation for λ, and can be solved (in theory) with the Green's function of the d'Alembertian. Since the existence and uniqueness of a solution is guaranteed, the gauge is considered viable and we can proceed. The potentials in the Lorenz gauge are then recovered as:

[6] As we will see later, this is actually still the ideal choice of gauge for quantisation. We will just have to remove the inhomogeneities so it reduces to a simple wave equation.

$$\Box V(\mathbf{r},t) = -\frac{1}{\epsilon_0}\rho(\mathbf{r},t) \tag{1.20}$$

$$\Box \mathbf{A}(\mathbf{r},t) = -\mu_0 \mathbf{J}(\mathbf{r},t) \tag{1.21}$$

These are simply inhomogeneous wave equations for the scalar and vector potentials, with a great degree of symmetry in the treatment! Once again, these are soluble by the Green's function (in theory). This gauge proves particularly useful for relativistic and radiative calculations, since it places V/c and $\mathbf{A}$ on equal footing. In this gauge, the remaining two Maxwell's equations also reduce to a particularly neat form:

$$\partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu \tag{1.22}$$

where $j^\mu = (c\rho, \mathbf{J})$ is the 4-current, and the Greek indices run over the values $\mu = 0, 1, 2, 3$.

1.2.3 The R_ξ Gauge

As a little extra tidbit, we also consider the generalised Lorenz gauge, most commonly known as the R_ξ gauge. The gauge invariance of electrodynamics occurs at a more fundamental level than what has been shown here – it appears in the Lagrangian of electrodynamics. Since it demonstrates a continuous symmetry under gauge transformations, it appears that there must be some sort of conserved quantity, but this is not true in reality. Noether's theorem states that only *global symmetries* admit a conservation law, while a gauge symmetry is a far weaker condition and instead represents a local symmetry (of which the global symmetry is a special case)[7]. The quantisation of a field described by a gauge-invariant Lagrangian gives rise to **gauge bosons**, as they are commonly known. In these situations, instead of fixing the gauge *a priori* via the imposition of a gauge constraint, we add a gauge-breaking term to the Lagrangian which breaks this local symmetry. In the case of the R_ξ gauge, this gauge-breaking term is:

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} + \delta\mathcal{L} = \mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{1.23}$$

The Lorenz gauge from above, or Landau gauge in this context, is a special case of this where we take the limit $\xi \to 0$ after quantising the field. In quantum electrodynamics, the most commonly used gauge is the Feynman-t'Hooft gauge in which $\xi = 1$[8].

1.3 Gauge Choice for Quantum Optics

The whole point of gauge fixing was to reduce the number of degrees of freedom as far as possible. In moving from the fields $(\mathbf{E}, \mathbf{B})$ to the potentials $(V, \mathbf{A})$, we went from six to four degrees of freedom. In the Lorenz gauge, we obtained symmetric equations for the scalar and vector potentials. While elegant, this still requires the specification of four values at each point in spacetime to fully characterise the fields (even in the absence of sources, since this is akin to a (3+1)-dimensional Laplace's equation). Thus, the Lorenz gauge is not our ideal choice.

Instead, in the Coulomb gauge, the imposition of no sources results in the following:

[7] See Noether's second theorem.

[8] If none of this makes sense at this point, that is perfectly fine. You can return to this after we have completed the quantisation in our quantum optics context to revisit this idea.

$$\nabla^2 V = 0 \overset{\rho=0}{\Longrightarrow} V(\mathbf{r}, t) = 0 \tag{1.24}$$

$$\Box \mathbf{A} = 0 \tag{1.25}$$

where in the second equation, both the source term $\mathbf{J}$ and scalar potential term $\dot{V}$ vanish, producing a homogeneous wave equation for $\mathbf{A}$. Thus, we only need three degrees of freedom in the Coulomb gauge, which is far easier to deal with! In this gauge, the general solution is easily written as:

$$\mathbf{A}(\mathbf{r}, t) = A_0 \hat{\varepsilon} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \quad \text{where} \quad \mathbf{k} \cdot \hat{\varepsilon} = 0 \tag{1.26}$$

The imposition of the Coulomb gauge from Eq. 1.13 leads to the condition that the wavevector $\mathbf{k}$ must be orthogonal to the polarisation vector $\hat{\varepsilon}$, which implies that $\mathbf{A}$ is now a purely transverse field. For this reason, the Coulomb gauge is often called the transverse gauge in the field of quantum optics. This plane wave form of the vector potential is incredibly useful, since we know the Fourier basis is complete over $\mathbb{R}^3$, and so the vector potential within a region of space can be decomposed into Fourier modes using these plane waves. For instance, if we take a cavity of side length L, the vector potential within the cavity can be arbitrarily decomposed as:

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{k}} \sum_{\mu} A_{\mathbf{k},\mu} \hat{\varepsilon}_\mu e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)} \tag{1.27}$$

where we have summed over cavity modes (wavevectors) $\mathbf{k}$ and polarisations $\mu = 1, 2$ for the two orthogonal transverse polarisations, with coefficients $A_{\mathbf{k},\mu}$.

2 Canonical Variables of the Electromagnetic Field

There is nothing quantum about any of what we have done above, including the Fourier decomposition of the vector potential. To truly move into a quantised regime, we have to impose some form of **canonical commutation relation**, akin to the fundamental commutator of quantum mechanics. The fundamental canonical commutation relation is postulated (or obtained through the correspondence principle from the canonical Poisson bracket) as:

$$[q_i, p_j] = i\hbar \delta_{ij} \tag{2.1}$$

To carry out the canonical quantisation procedure here, we need to find two canonically conjugate variables that obey the symplectic Hamiltonian structure[9]:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.2}$$

where the generalised coordinates and generalised momenta $\{q_i, p_i\}$ form conjugate variable pairs. From classical mechanics, we know that the simplest way to obtain the canonical momentum conjugate p_i for a generalised coordinate q_i is through:

[9] Do not worry if the appearance of the term symplectic is new to you here, as it is not necessary for understanding this content. The phase space of generalised coordinates and momenta in which particles live is known as a symplectic manifold M, equipped with a tensor field known as a closed differential 2-form ω. This 2-form is constructed from an exterior algebra on the manifold involving the generalised coordinates and momenta, and naturally obeys Hamilton's equations by generating a flow field on the manifold. This is the purely geometric view of classical Hamiltonian evolution of a system through phase space. If you are interested in differential geometry and mathematical physics, you can look these terms up.

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{2.3}$$

Thus, if we consider the 4-potential to be the generalised coordinate, we need the Lagrangian of the EM field to obtain a generalised momentum conjugate! The following section is very mathematically involved, and assumes knowledge of tensor calculus up to the use of the metric tensor. If this is foreign to you, feel free to skip the next section as it will not affect your understanding of the quantisation procedure. This is really just here for completeness, and for my own bookkeeping.

2.1 Interlude: Classical Field Theory and the EM Lagrangian

I will now state without proof[10] that the Lagrangian density[11] of the EM field is given by [3]:

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \tag{2.4}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the EM field tensor (Eq. 12.119 in Griffiths). The two quantities $(A_\mu, \partial_\nu A_\mu)$ are the generalised field and derivative, a field-theoretic version of the generalised coordinate and velocity usually written as $(q_i, \dot{q}_i)$. To confirm that this Lagrangian is sound, let's obtain the equations of motion for the 4-potential and check that it agrees with Maxwell's equations. The field theoretic analogue of the Euler-Lagrange equation is written:

$$\partial_\alpha \left[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} \right] - \frac{\partial \mathcal{L}}{\partial A_\beta} = 0 \tag{2.5}$$

It is simpler to tackle this term-by-term, so we begin by computing the derivative of the Lagrangian with respect to the generalised field derivative, which can be rather involved:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} &= -\frac{1}{4\mu_0} \frac{\partial}{\partial(\partial_\alpha A_\beta)} \left(F_{\mu\nu} g^{\lambda\mu} g^{\sigma\nu} F_{\lambda\sigma} \right) \\
&= -\frac{1}{4\mu_0} g^{\lambda\mu} g^{\sigma\nu} \left[F_{\mu\nu} \frac{\partial F_{\lambda\sigma}}{\partial(\partial_\alpha A_\beta)} + F_{\lambda\sigma} \frac{\partial F_{\mu\nu}}{\partial(\partial_\alpha A_\beta)} \right] \\
&= -\frac{1}{4\mu_0} g^{\lambda\mu} g^{\sigma\nu} \left[F_{\mu\nu} \left(\delta^\alpha_\lambda \delta^\beta_\sigma - \delta^\beta_\lambda \delta^\alpha_\sigma \right) + F_{\lambda\sigma} \left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu \right) \right] \\
&= -\frac{1}{4\mu_0} \left(F^{\alpha\beta} - F^{\beta\alpha} + F^{\alpha\beta} - F^{\beta\alpha} \right) \\
&= -\frac{1}{\mu_0} F^{\alpha\beta}, \quad \text{by the antisymmetry of } F^{\alpha\beta}
\end{aligned}$$

The second term in Eq. 2.5 is simply:

$$\frac{\partial \mathcal{L}}{\partial A_\beta} = -\delta^\beta_\mu j^\mu = -j^\beta$$

Combining these gives us the field equations of motion as:

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{2.6}$$

Explicitly expanding the tensor indices, we can see that the equation for $\nu = 0$ returns Gauss' law, while the equations for $\nu = 1, 2, 3$ return the three components of Ampere's law! The remaining two Maxwell's

[10] The proof of this is honestly rather lengthy, and requires a lot of physical intuition in classical field theory which does not aid the ongoing discussion. For better clarity, refer to Chapter 5 of [3] and Chapter 1 of [4].

[11] This is a density, so the Lagrangian L must be obtained by integrating this density over all space.

equations can be obtained from the Hodge dual to the EM field tensor (Eq. 12.120 of Griffiths), which I will not demonstrate here. Now that we have confirmed that the Lagrangian agrees with Maxwell's equations, we can extract our desired conjugate momentum field to A_μ as:

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} \tag{2.7}$$

To see what this conjugate momentum field really is, we can explicitly expand the Lagrangian from Eq. 2.4 and perform the derivatives to obtain the components of this momentum field. We proceed with the expansion as:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu \\
&= -\frac{1}{4\mu_0}\left(2F_{0\nu}F^{0\nu} + F_{ij}F^{ij}\right) - A_\mu j^\mu \\
&= -\frac{1}{2\mu_0}\left[\left(\frac{1}{c}\dot{A}^i\right)^2 + \left(\boldsymbol{\nabla} A^0\right)^2 - \frac{2}{c^2}\dot{A}^i \partial^i A_0\right] - \frac{1}{4\mu_0}F_{ij}F^{ij} - A_\mu j^\mu
\end{aligned}$$

where the Latin indices run over $i = 1, 2, 3$ to denote that they are solely for the spatial part. From this expansion, we thus obtain the explicit components of the conjugate momentum as ($A_\mu, \dot{A}_\mu, \partial_\alpha A_\mu$ are all independent variables here):

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{A}^0} = 0 \tag{2.8}$$

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = \frac{1}{\mu_0 c^2}\dot{A}^i = -\epsilon_0 E_i \tag{2.9}$$

While it seems strange that the temporal component of the conjugate momentum identically vanishes, this actually fits into our gauge constraint where we found that $\partial V/\partial t = 0$ based on the absence of a source distribution. Since the scalar potential vanishes, it only seems reasonable that the Lagrangian cannot possibly depend on it or any of its derivatives, so the temporal component of the momentum field vanishes. We thus have our conjugate variable pair as (A_i, Π_i), and can proceed with the quantisation!

Since the process of imposing a canonical commutation relation requires the system to be expressed in terms of the canonical coordinates and their conjugate momenta, it is only natural for us to now transform the Lagrangian (whose canonical variables are the field and its derivatives) into the Hamiltonian (whose canonical variables are the field and its conjugate momentum). For this, we perform a Legendre transformation on the Lagrangian density to convert the variable A_i into Π_i, obtaining the Hamiltonian density as:

$$\mathcal{H}(A_i, \Pi_i) = \Pi_i \dot{A}_i(A_i, \Pi_i) - \mathcal{L}(A_i, \dot{A}_i(A_i, \Pi_i)) \tag{2.10}$$

$$= \frac{1}{2}\epsilon_0\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) - A_0(\boldsymbol{\nabla}\cdot\mathbf{E}) \tag{2.11}$$

where the second equality takes some effort to obtain, but is a rather straightforward manipulation of the terms in the Lagrangian (assuming no sources such that $j^\mu = 0$). Thus, we have recovered the expected energy density of the EM field! we see that A_0 is a Lagrange multiplier which imposes Gauss' law, which should trivially return zero in the absence of any sources. Before proceeding any further, we have to first deal with an annoyance that could potentially ruin the process. The canonical variables (A_i, Π_i) are vector fields, which are intrinsically difficult to impose canonical commutation relations on. We should thus try to obtain scalars from

these vector fields, upon which we can impose the commutation relation to achieve a quantised theory. For this, we first decompose the canonical variables into their spatial Fourier modes:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{k}\in\mathcal{G}} \tilde{\mathbf{A}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.12}$$

$$\mathbf{\Pi}(\mathbf{r}) = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{k}\in\mathcal{G}} \tilde{\mathbf{\Pi}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.13}$$

Here, the vectorial coefficients $\{\tilde{\mathbf{A}}_{\mathbf{k}}, \tilde{\mathbf{\Pi}}_{\mathbf{k}}\}$ are the mode amplitudes for each of the modes with wavevectors $\{\mathbf{k}\}$. Since we have considered a finite cavity, the usual boundary conditions of vanishing amplitude at the boundaries set a discretisation condition on the wavevectors, namely that $\mathbf{k} = \frac{2\pi}{L}(n_x, n_y, n_z) \in \mathcal{G}$, with $(n_x, n_y, n_z) \in \mathbb{Z}^3$. In the limit of $L \longrightarrow \infty$, the Fourier $\mathbf{k}$-space $\mathcal{G}$ effectively maps to $\mathbb{R}^3$. All Fourier modes here are also orthonormal due to the condition:

$$\int \mathrm{d}\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}'\cdot\mathbf{r}} = L^3 \delta_{\mathbf{k},\mathbf{k}'} \tag{2.14}$$

2.2 Decoupling the Canonical Variables

In the previously mentioned mode decomposition, we can also reveal another problem in our choice of canonical variables. The quantities $\mathbf{A}$ and $\mathbf{\Pi}$ must be real (otherwise the EM fields would be complex, which is unphysical), so this forces a constraint on the mode amplitudes. To determine this constraint, we employ a stupid trick:

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{1}{\sqrt{L^3}} \sum_{\mathbf{k}\in\mathcal{G}} \tilde{\mathbf{A}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \frac{1}{2\sqrt{L^3}} \sum_{\mathbf{k}\in\mathcal{G}} \left(\tilde{\mathbf{A}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + \tilde{\mathbf{A}}_{-\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} \right)\end{aligned}$$

where we split the term into two and take $\mathbf{k} \to -\mathbf{k}$ in the second term. To impose that $\mathbf{A}(\mathbf{r}) \in \mathbb{R}$, we need each term in the parentheses to identically be real (regardless of $\mathbf{k}$). For the resulting term to be real, the condition on the mode amplitudes is thus:

$$\tilde{\mathbf{A}}^*_{\mathbf{k}} = \tilde{\mathbf{A}}_{-\mathbf{k}} \tag{2.15}$$

Simply put, the amplitude of the $-\mathbf{k}$ mode must be the complex conjugate of the $\mathbf{k}$ mode. Plugging this into the above mode decomposition, we obtain:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2\sqrt{L^3}} \sum_{\mathbf{k}\in\mathcal{G}} \left(\tilde{\mathbf{A}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + \tilde{\mathbf{A}}^*_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} \right) \tag{2.16}$$

with a similar constraint and expression for the conjugate momentum field. Due to these constraints on $\tilde{\mathbf{A}}_{\mathbf{k}}$ and $\tilde{\mathbf{\Pi}}_{\mathbf{k}}$, we instead define a new variable that is free from this nonsense:

$$\tilde{\alpha}_{\mathbf{k}} = \frac{1}{h(\mathbf{k})} \left[\epsilon_0 \omega(\mathbf{k}) \tilde{\mathbf{A}}_{\mathbf{k}} + i \tilde{\mathbf{\Pi}}_{\mathbf{k}} \right], \quad \text{where} \quad \omega(\mathbf{k}) = c|\mathbf{k}| \tag{2.17}$$

where $h(\mathbf{k})$ is an arbitrary scalar that is even in $\mathbf{k}$. This variable definition seems incredibly arbitrary, and in all honesty it is. However, it is an incredibly useful quantity that we can form out of $\tilde{\mathbf{A}}_{\mathbf{k}}$ and $\tilde{\mathbf{\Pi}}_{\mathbf{k}}$, which frees us from the constraints nested within the initial variables. Consider the following:

$$\begin{aligned}\tilde{\alpha}^*_{-\mathbf{k}} &= \frac{1}{h(\mathbf{k})}\left[\epsilon_0\omega(\mathbf{k})\tilde{\mathbf{A}}^*_{-\mathbf{k}} - i\tilde{\mathbf{\Pi}}^*_{-\mathbf{k}}\right] \\ &= \frac{1}{h(\mathbf{k})}\left[\epsilon_0\omega(\mathbf{k})\tilde{\mathbf{A}}_{\mathbf{k}} - i\tilde{\mathbf{\Pi}}_{\mathbf{k}}\right] \neq \tilde{\alpha}_{\mathbf{k}}\end{aligned}$$

Thus, the constraints manifest in $\tilde{\mathbf{A}}_{\mathbf{k}}$ and $\tilde{\mathbf{\Pi}}_{\mathbf{k}}$ are no longer present in $\tilde{\alpha}_{\mathbf{k}}$! However, this is still a vectorial quantity, so we're not quite ready to call $\tilde{\alpha}_{\mathbf{k}}$ and $\tilde{\alpha}^*_{\mathbf{k}}$ conjugate variables for quantisation yet. For one, we have yet to impose the Coulomb gauge on this new variable. Imposing the Coulomb gauge tells us that $\tilde{\mathbf{A}}_{\mathbf{k}}$ is orthogonal to $\mathbf{k}$ (transverse), while Gauss' law in the absence of charges tells us that $\tilde{\mathbf{\Pi}}_{\mathbf{k}}$ is also transverse. Combining these two based on the definition of $\tilde{\alpha}_{\mathbf{k}}$ tells us that $\mathbf{k}\cdot\tilde{\alpha}_{\mathbf{k}} = 0$. Thus, we see that $\tilde{\alpha}_{\mathbf{k}}$ is also transverse and must also have two available polarisations. Accordingly, we define:

$$\tilde{\alpha}_{\mathbf{k}} = \sum_{p=1,2}\alpha_{\mathbf{k},p}\hat{\mathbf{e}}_p, \quad \text{with} \quad \hat{\mathbf{e}}_p\cdot\hat{\mathbf{e}}_{p'} = \delta_{pp'} \tag{2.18}$$

Finally, the independent scalar variables that we have been looking for have arrived in the form of the pair $(\alpha_{\mathbf{k},p}, \alpha^*_{\mathbf{k},p})$. Now, we just have to re-express both $\tilde{\mathbf{A}}_{\mathbf{k}}$ and $\tilde{\mathbf{\Pi}}_{\mathbf{k}}$ in terms of these independent scalars, which turns out to be quite the mathematical mess. As I'm not willing to transcribe the process step by step[12], here is the general procedure.

1. Start with the mode decomposition from Eqs. 2.12 and 2.13, since it's still perfectly valid.
2. Define $h(\mathbf{k}) = \sqrt{2\epsilon_0\omega(\mathbf{k})\hbar}$ as the arbitrary scalar in the definition of $\tilde{\alpha}_{\mathbf{k}}$.
3. Write $\tilde{\mathbf{A}}_{\mathbf{k}}$ and $\tilde{\mathbf{\Pi}}_{\mathbf{k}}$ in terms of $\alpha_{\mathbf{k},p}$ and $\alpha^*_{\mathbf{k},p}$.
4. Plug everything back into Eqs. 2.12 and 2.13, simplifying with the constraint from Eq. 2.15.

If all of this is done correctly, you should arrive at the following expressions for $\tilde{\mathbf{A}}_{\mathbf{k}}$ and $\tilde{\mathbf{\Pi}}_{\mathbf{k}}$ in terms of the pair $(\alpha_{\mathbf{k},p}, \alpha^*_{\mathbf{k},p})$:

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}\in\mathcal{G}}\sum_p \alpha_{\mathbf{k},p}\mathcal{A}_{\mathbf{k},p} + \text{c.c}, \quad \text{where} \quad \mathcal{A}_{\mathbf{k},p} = \sqrt{\frac{\hbar}{2\epsilon_0\omega(\mathbf{k})L^3}}e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{e}}_p \tag{2.19}$$

$$\mathbf{\Pi}(\mathbf{r}) = \sum_{\mathbf{k}\in\mathcal{G}}\sum_p -i\epsilon_0\omega(\mathbf{k})\alpha_{\mathbf{k},p}\mathcal{A}_{\mathbf{k},p} + \text{c.c} \tag{2.20}$$

From these two initial variables, the electric and magnetic fields are also quite easily derived as:

$$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k}\in\mathcal{G}}\sum_p i\omega(\mathbf{k})\alpha_{\mathbf{k},p}\mathcal{A}_{\mathbf{k},p} + \text{c.c} \tag{2.21}$$

$$\mathbf{B}(\mathbf{r}) = \sum_{\mathbf{k}\in\mathcal{G}}\sum_p i\mathbf{k}\times\alpha_{\mathbf{k},p}\mathcal{A}_{\mathbf{k},p} + \text{c.c} \tag{2.22}$$

Using the orthogonality of the modes, we can use the inverse Fourier decomposition to rewrite these scalars in terms of the real initial variables in position space:

$$\alpha_{\mathbf{k},p} = \frac{1}{\hbar}\int \mathrm{d}\mathbf{r}\,\mathcal{A}^*_{\mathbf{k},p}\cdot[\epsilon_0\omega(\mathbf{k})\mathbf{A}(\mathbf{r}) + i\mathbf{\Pi}(\mathbf{r})] \tag{2.23}$$

[12]The real story is that I lost the napkins that I did this derivation on at a sushi bar and have since been too lazy to reproduce this, but I promise you the math checks out.

Finally, we have all the pieces in place and we can start making meaningful steps toward quantisation. With all of our initial variables and fields written in terms of these independent scalars, the effects of the quantisation procedure on the scalars can be methodically extended to our initial problem.

3 The Hamiltonian Formalism

Now, because I genuinely have no real way to motivate this[13], *consider the following*:

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{G}}\sum_{p}\hbar\omega(\mathbf{k})|\alpha_{\mathbf{k},p}|^2 &= \sum_{\mathbf{k}\in\mathcal{G}}\sum_{p}\omega(\mathbf{k})\int \mathrm{d}\mathbf{r}\,\alpha^*_{\mathbf{k},p}\mathcal{A}^*_{\mathbf{k},p}\cdot[\epsilon_0\omega(\mathbf{k})\mathbf{A}(\mathbf{r})+i\mathbf{\Pi}(\mathbf{r})] \\ &= \int \mathrm{d}\mathbf{r}\left\{\mathbf{A}(\mathbf{r})\cdot\left[\epsilon_0\sum_{\mathbf{k},p}\alpha^*_{\mathbf{k},p}\omega(\mathbf{k})^2\mathcal{A}^*_{\mathbf{k},p}\right]+\mathbf{\Pi}(\mathbf{r})\cdot\left[i\sum_{\mathbf{k},p}\alpha^*_{\mathbf{k},p}\mathcal{A}^*_{\mathbf{k},p}\right]\right\} \end{aligned} \tag{3.1}$$

where in the second line, I have condensed the two sums into one since it is clear that $\mathbf{k}$ must be an element of the discrete $\mathbf{k}$-space $\mathcal{G}$. To massage this expression further, we consider the definition of $\mathcal{A}_{\mathbf{k},p}$ and act on it with the Laplacian to obtain:

$$\omega(\mathbf{k})^2\mathcal{A}_{\mathbf{k},p} = c^2\nabla^2\mathcal{A}_{\mathbf{k},p} \tag{3.2}$$

which can be used to tame the first term in Eq. 3.1, after which it reduces to Eq. 2.19 by comparison[14]. For the second term, direct comparison with Eq. 2.20 reveals that it is simply equal to the complex conjugate of the momentum field. We can thus write:

$$\sum_{\mathbf{k}\in\mathcal{G}}\sum_{p}\hbar\omega(\mathbf{k})|\alpha_{\mathbf{k},p}|^2 = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,\epsilon_0 c^2\mathbf{A}(\mathbf{r})\cdot\left[\nabla^2\mathbf{A}(\mathbf{r})\right]+\frac{1}{2}\int \mathrm{d}\mathbf{r}\,\frac{1}{\epsilon_0}|\mathbf{\Pi}(\mathbf{r})|^2 \tag{3.3}$$

From the definition of the conjugate momentum field in Eq. 2.9, the second term is quite clearly equal to

$$\frac{\epsilon_0}{2}\int \mathrm{d}\mathbf{r}\,|\mathbf{E}(\mathbf{r})|^2$$

, which is the total energy stored in the electric field! Turning our attention to the first term now, we proceed as:

$$\begin{aligned}\mathbf{A}\cdot\nabla^2\mathbf{A} &= \mathbf{A}\cdot[\boldsymbol{\nabla}(\cancel{\boldsymbol{\nabla}\cdot\mathbf{A}})-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})] \quad \text{vector Laplacian identity} \\ &= -\mathbf{A}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) \quad \text{using the Coulomb gauge} \\ &= -(\boldsymbol{\nabla}\times\mathbf{A})^2+\boldsymbol{\nabla}\cdot[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A})] \quad \text{let } \mathbf{X}=\boldsymbol{\nabla}\times\mathbf{A}\text{, then use } \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{X})\end{aligned}$$

Under the integral over all space, this expression can be dealt with as follows:

$$\begin{aligned}\int_{\mathcal{V}}\mathrm{d}\mathbf{r}\,\mathbf{A}\cdot\nabla^2\mathbf{A} &= -\int \mathrm{d}\mathbf{r}\,(\boldsymbol{\nabla}\times\mathbf{A})^2+\int_{\mathcal{V}}\mathrm{d}\mathbf{r}\,\boldsymbol{\nabla}\cdot[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A})] \\ &= -\int \mathrm{d}\mathbf{r}\,(\boldsymbol{\nabla}\times\mathbf{A})^2+\cancel{\int_{\partial\mathcal{V}}\mathrm{d}\mathbf{S}\,[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{A})]}\end{aligned}$$

[13] In theory, one could perform the forward version of this proof, but that is simply too elusive for me at this stage.

[14] Note that the Laplacian operator can be pulled out to act on the entirety of $\mathbf{A}(\mathbf{r})$ since all spatial dependence is solely within $\mathcal{A}_{\mathbf{k},p}$.

where the second term vanishes since $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$ vanishes at the boundaries (in the infinite limit). The first term should look familiar, and we can now put this back in Eq. 3.3 to finally obtain:

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{G}}\sum_{p}\hbar\omega(\mathbf{k})\alpha^*_{\mathbf{k},p}\alpha_{\mathbf{k},p} &= \frac{1}{2}\int \mathrm{d}\mathbf{r}\left[\epsilon_0\mathbf{E}(\mathbf{r})^2 + \frac{1}{\mu_0}\mathbf{B}(\mathbf{r})^2\right] \\ &= \int \mathrm{d}\mathbf{r}\,\mathcal{H} = H \end{aligned} \tag{3.4}$$

which is the total energy, or the Hamiltonian of the EM field! Now that we have reconciled our work with something that we are familiar with, let's turn our attention to the left hand side (LHS) that we began with. This looks an awful lot like the Hamiltonian of the quantum harmonic oscillator, with the only difference being that we have scalars $(\alpha_{\mathbf{k},p}, \alpha^*_{\mathbf{k},p})$ instead of the expected creation and annihilation operators $(a, a^\dagger)$. While this is exactly the track we want to be on, it is worth noting that we have forgotten to look at half of the problem. Specifically, everything up until this point has dealt with the spatial portion of the vector potential and its conjugate momentum field, ignoring the time evolution of the fields!

3.1 Time Evolution in the Hamiltonian Formalism

Now that we have the spatial part of our scalar variables, we should study the evolution of the temporal part to see if it simplifies the problem any further or provides some valuable insight. We first write:

$$\dot{\alpha}_{\mathbf{k},p} = \frac{1}{\hbar}\int \mathrm{d}\mathbf{r}\,\mathcal{A}^*_{\mathbf{k},p}\cdot\left[\epsilon_0\omega(\mathbf{k})\dot{\mathbf{A}}(\mathbf{r}) + i\dot{\mathbf{\Pi}}(\mathbf{r})\right] \tag{3.5}$$

Here, we have to deal with the time derivatives of the two initial canonical variables, which is thankfully not too difficult with some basic arguments. From the definition of the conjugate momentum field in Eq. 2.9, we know that $\dot{\mathbf{A}} \sim \mathbf{\Pi}/\epsilon_0$. Furthermore, this also implies that $\dot{\mathbf{\Pi}} \sim \epsilon_0\ddot{\mathbf{A}}$. Then, from the wave equation for $\mathbf{A}$ in Eq. 1.25, we easily see that $\ddot{\mathbf{A}} \sim \epsilon_0 c^2\nabla^2\mathbf{A}$. Combining these two pieces into Eq. 3.5, we obtain:

$$\dot{\alpha}_{\mathbf{k},p} = \frac{1}{\hbar}\int \mathrm{d}\mathbf{r}\,\mathcal{A}^*_{\mathbf{k},p}\cdot\left[\omega(\mathbf{k})\mathbf{\Pi}(\mathbf{r}) + i\epsilon_0 c^2\nabla^2\mathbf{A}(\mathbf{r})\right] \tag{3.6}$$

Let's now focus on the second term, which still contains a spatial derivative. We can deal with this by invoking Green's vector identity:

$$\mathbf{P}\cdot\nabla^2\mathbf{Q} - \mathbf{Q}\cdot\nabla^2\mathbf{P} = \boldsymbol{\nabla}\cdot[\mathbf{P}(\boldsymbol{\nabla}\cdot\mathbf{Q}) - \mathbf{Q}(\boldsymbol{\nabla}\cdot\mathbf{P})] \tag{3.7}$$

With this, the second term can be written as:

$$\begin{aligned}\int_{\mathcal{V}}\mathrm{d}\mathbf{r}\,\mathcal{A}^*_{\mathbf{k},p}\cdot\nabla^2\mathbf{A} &= \int_{\mathcal{V}}\mathrm{d}\mathbf{r}\,\mathbf{A}\cdot\nabla^2\mathcal{A}^*_{\mathbf{k},p} + \int_{\mathcal{V}}\boldsymbol{\nabla}\cdot\left[\mathcal{A}^*_{\mathbf{k},p}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \mathbf{A}\left(\boldsymbol{\nabla}\cdot\mathcal{A}^*_{\mathbf{k},p}\right)\right] \\ &= -|\mathbf{k}|^2\int_{\mathcal{V}}\mathrm{d}\mathbf{r}\,\mathbf{A}\cdot\mathcal{A}^*_{\mathbf{k},p} + \cancel{\int_{\partial\mathcal{V}}\mathrm{d}\mathbf{S}\cdot\left[\mathcal{A}^*_{\mathbf{k},p}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \mathbf{A}\left(\boldsymbol{\nabla}\cdot\mathcal{A}^*_{\mathbf{k},p}\right)\right]}\end{aligned}$$

where I have used Stokes' theorem on the second term, followed by the cancellation using intuition that $\mathbf{A}$ vanishes on the boundary. Finally, we can insert this into Eq. 3.6 to obtain a differential equation for the scalar variables:

$$\begin{aligned}\dot{\alpha}_{\mathbf{k},p} &= \frac{1}{\hbar}\int \mathrm{d}\mathbf{r}\,\mathcal{A}^*_{\mathbf{k},p}\cdot\left[\omega(\mathbf{k})\mathbf{\Pi}(\mathbf{r}) - i\epsilon_0\omega(\mathbf{k})^2\mathbf{A}(\mathbf{r})\right] \\ &= -i\omega(\mathbf{k})\alpha_{\mathbf{k},p}\end{aligned} \tag{3.8}$$

This is a rather trivial equation to solve, and we can do so by inspection to finally write the time evolution of each $\alpha_{\mathbf{k},p}$ amplitude as:

$$\alpha_{\mathbf{k},p}(t) = \alpha_{\mathbf{k},p}(0)e^{-i\omega(\mathbf{k})t} \tag{3.9}$$

As simple as it sounds, the physical implication of this statement is surprisingly deep. We have just shown that the evolution of every mode amplitude is independent of any other mode, which is commonly known as the absence of mode coupling. Earlier, I had pointed out that intuition is pushing us toward quantising the EM field using a harmonic oscillator formalism. We can imagine all of space to be comprised of a grid of points, each point containing an oscillator at every possible mode for the EM field. The motion of any one oscillator is coupled to all spatially adjacent oscillators, but only those of the same mode. This information is not new, and was already contained within Maxwell's equations. The new insight here is that the motion of one oscillator at a given spatial point will not trigger the motion of other oscillators at the same point of different modes. This is a more intuitive reason behind why a light wave of frequency 2ω does not spontaneously decay into two simultaneous waves of frequency ω each. With this intuition in place, we are finally ready to proceed with the actual quantisation procedure.

4 Canonical Quantisation of the Electromagnetic Field

Now that the theatrics have passed and we know full well that we are heading in the direction of a harmonic oscillator, it is perhaps worth the short detour to review the canonical quantisation procedure for a classical harmonic oscillator. This will shed some light on how to approach the quantisation of the independent scalar variables that we obtained before, as well as provide a nice segue into the later topic of the actual canonical variables of the photon field.

4.1 Interlude: Canonical Quantisation of the Harmonic Oscillator

From introductory classical mechanics, we know that the Hamiltonian of the classical oscillator is written as:

$$H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2, \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}} \text{ and } k = \left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0} \tag{4.1}$$

where I have used q to denote the position and p to denote the momentum. From Hamilton's equations in Eq. 2.2, we can then obtain the equations of motion as:

$$\begin{cases} \dot{q} &= \frac{1}{m}p \\ \dot{p} &= -m\omega^2 q \end{cases} \tag{4.2}$$

This is perfectly fine, but it'd be ideal to have a picture of this that doesn't care about the oscillator's mass. To rid ourselves of this, we perform a canonical transformation [5] to rescale the coordinates:

$$\begin{cases} p \longrightarrow P = p/\sqrt{m} \\ q \longrightarrow Q = q\sqrt{m} \end{cases} \tag{4.3}$$

In these new coordinates, the Hamiltonian and resulting equations of motion are:

$$H(Q,P) = \frac{1}{2}\left(P^2 + \omega^2 Q^2\right) \Longrightarrow \begin{cases} \dot{Q} &= P \\ \dot{P} &= -\omega^2 Q \end{cases} \tag{4.4}$$

Finally, we perform one last stupid trick to obtain:

$$\begin{aligned} H(Q,P) &= \frac{1}{2}\left(Q^2+\omega^2P^2\right) \\ &= \left(\frac{\omega Q+iP}{\sqrt{2}}\right)\left(\frac{\omega Q-iP}{\sqrt{2}}\right) \\ &= h_0\omega\beta^*\beta, \quad \text{where} \quad \beta = \frac{\omega Q+iP}{\sqrt{2h_0\omega}} \end{aligned} \tag{4.5}$$

where h_0 is just a constant. Everything done here so far has been classical, but this gives us a beautiful link between the Hamiltonian of Eq. 3.4 and the familiar classical coordinates of phase space. In canonical quantisation, the classical Poisson bracket $\{q,p\}$ transcends the mortal coil and attains the status of being a commutator as $\frac{1}{i\hbar}[q,p]$. To proceed with quantising the classical oscillator, we have to promote the scalars (β,β^*) into operators, and in doing so we break a key assumption that they (and the underlying coordinates (q,p)) commute. We promote them to operators as:

$$\begin{cases} \beta & \longrightarrow \hat{\beta} = \frac{\omega\hat{q}+i\hat{p}}{\sqrt{2\hbar\omega}} \\ \beta^* & \longrightarrow \hat{\beta}^\dagger = \frac{\omega\hat{q}-i\hat{p}}{\sqrt{2\hbar\omega}} \end{cases} \tag{4.6}$$

where the constant h_0 turns into the reduced Planck's constant $\hbar$ upon quantisation (surprise). Now, we consider the following (dropping the hat notation on operators since it's clear things are quantum from here):

$$\begin{aligned} \beta^\dagger\beta &= \frac{1}{2\hbar\omega}(\omega q-ip)(\omega q+ip) \\ &= \frac{1}{2\hbar\omega}\left(\omega^2q^2+p^2+i\omega[q,p]\right) \end{aligned}$$

We now invoke the canonical commutation relation $[q,p]=i\hbar\mathbb{1}$, and return to the Hamiltonian:

$$\begin{aligned} H &= \hbar\omega\beta^\dagger\beta+\frac{1}{2}\hbar\omega\mathbb{1} \\ &= \hbar\omega\left(\beta^\dagger\beta+\frac{1}{2}\mathbb{1}\right) \\ &\equiv \hbar\omega\left(a^\dagger a+\frac{1}{2}\right) \end{aligned} \tag{4.7}$$

where it is clear that the operators $(\beta^\dagger,\beta)$ were just the usual raising and lowering operators $(a^\dagger,a)$ all along. I assume none of this is new to anyone who has previously taken an introductory quantum mechanics course, so I'll skip the derivations for the properties of this Hamiltonian. Of greatest importance is that we recall that the eigenstates of the quantum harmonic oscillator are the number states $\{|n\rangle\}$, obtained by repeated action of the raising operator on the ground state:

$$|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle \tag{4.8}$$

4.2 At Last, Photons

Finally, with the reconciliation of the EM field Hamiltonian and its similarity to the quantum harmonic oscillator Hamiltonian, we can undertake the quantisation process that I promised many pages ago. Returning to Eq. 3.4, we can use the idea that all modes are independent and decoupled to write:

$$H = \sum_{\mathbf{k},p} \hbar\omega(\mathbf{k}) \alpha^*_{\mathbf{k},p} \alpha_{\mathbf{k},p} \equiv \sum_{\mathbf{k},p} H_{\mathbf{k},p} \tag{4.9}$$

We now carry out the quantisation within each $H_{\mathbf{k},p}$ by identifying each scalar $\alpha_{\mathbf{k},p}$ with a similar construction from Eq. 4.5, providing us with the variables to impose a canonical commutation relation as:

$$\alpha_{\mathbf{k},p} = \frac{\omega(\mathbf{k}) q_{\mathbf{k},p} + i p_{\mathbf{k},p}}{\sqrt{2\hbar\omega(\mathbf{k})}} \tag{4.10}$$

Imposing the canonical commutation relations on each of these conjugate variables involves promoting these variables to operators, so we define the raising and lowering operators of the EM field as:

$$\alpha_{\mathbf{k},p} \to a_{\mathbf{k},p} = \frac{\omega(\mathbf{k}) q_{\mathbf{k},p} + i p_{\mathbf{k},p}}{\sqrt{2\hbar\omega(\mathbf{k})}}, \quad \alpha^*_{\mathbf{k},p} \to a^\dagger_{\mathbf{k},p} = \frac{\omega(\mathbf{k}) q_{\mathbf{k},p} - i p_{\mathbf{k},p}}{\sqrt{2\hbar\omega(\mathbf{k})}} \tag{4.11}$$

From this, we finally obtain the quantised EM Hamiltonian in the form:

$$H_{\mathbf{k},p} = \hbar\omega(\mathbf{k}) \left(a^\dagger_{\mathbf{k},p} a_{\mathbf{k},p} + \frac{1}{2} \right) \tag{4.12}$$

While it may look like I have somehow magically converted the scalars $(\alpha_{\mathbf{k},p}, \alpha^*_{\mathbf{k},p})$ to operators $(a_{\mathbf{k},p}, a^\dagger_{\mathbf{k},p})$ and a second $+1/2$ term has dropped out of the sky, recall that the promotion of these variables to non-commuting operators absolutely allows for other terms to appear "out of nowhere" since they come from the commutator (which doesn't exist in the classical Hamiltonian). While it looks simple, this Hamiltonian for the EM field is incredibly powerful as it states that the energy of the EM field comes in discrete packets, each adding a discrete amount of energy $\hbar\omega(\mathbf{k})$ to the field's total energy. These discrete energy quanta are what we have come to know and love as **photons**.

Earlier, we established that the Hamiltonian takes the form of a sum over many independent contributions from the various modes. With this in mind, we can clearly see that the canonical quantisation of each mode forms independent elements in isolated Hilbert spaces, with the isolation coming from the fact that there is no coupling between the modes. However, we can still consider states that live in the **product space** of these mode spaces, which is formally known as a **Fock space**. This is written as:

$$\mathcal{F} = \bigotimes_{\mathbf{k} \in \mathcal{G}} \bigotimes_{p} \mathcal{H}_{\mathbf{k},p} \tag{4.13}$$

where $\mathcal{H}_{\mathbf{k},p}$ represents the Hilbert space for a single mode, and is isomorphic to the usual quantum harmonic oscillator Hilbert space of $L^2(\mathbb{C})$. Each $\mathcal{H}_{\mathbf{k},p}$ admits a complete basis $\{|n_{\mathbf{k},p}\rangle\}$, so the basis of the Fock space can be written as the tensor product of all individual bases from all modes. In the Fock space, the creation and annihilation operators are also generalised to:

$$\begin{aligned} a_{\mathbf{k}_j,p_j} &\equiv \mathbb{1}_{\mathbf{k}_1,p_1} \otimes \mathbb{1}_{\mathbf{k}_2,p_2} \otimes \cdots \otimes a_{\mathbf{k}_j,p_j} \otimes \mathbb{1}_{\mathbf{k}_{j+1},p_{j+1}} \otimes \cdots \\ a^\dagger_{\mathbf{k}_j,p_j} &\equiv \mathbb{1}_{\mathbf{k}_1,p_1} \otimes \mathbb{1}_{\mathbf{k}_2,p_2} \otimes \cdots \otimes a^\dagger_{\mathbf{k}_j,p_j} \otimes \mathbb{1}_{\mathbf{k}_{j+1},p_{j+1}} \otimes \cdots \end{aligned}$$

where j is a numerical index that counts all the modes. While this is countably infinite in the case of a finite cavity (the box of side length L that we started with), this becomes uncountably infinite in the limit of $L \to \infty$, so that will have to be treated carefully (beyond our scope). In this space, we also have the more general set of canonical commutation relations as:

$$[a_{\mathbf{k},p}, a_{\mathbf{k}',p'}] = \left[a^\dagger_{\mathbf{k},p}, a^\dagger_{\mathbf{k}',p'}\right] = 0 \tag{4.14}$$

$$\left[a_{\mathbf{k},p}, a^\dagger_{\mathbf{k}',p'}\right] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{p,p'}\mathbb{1} \tag{4.15}$$

These relations define a bosonic field which, together with the Lagrangian from Eq. 2.4, defines a massless spin-1 boson – the photon.

4.2.1 The Vacuum Energy

At this point, any celebrations should be promptly cut short by sirens and alarm bells going off at the apperance of Eq. 4.12. Specifically, we consider the vacuum eigenstate of this Hamiltonian and compute its energy:

$$\begin{aligned}\langle 0|H|0\rangle &= \sum_{\mathbf{k},p} \langle 0|H_{\mathbf{k},p}|0\rangle \\ &= \sum_{\mathbf{k},p} \hbar\omega(\mathbf{k})\left(\langle 0|a^\dagger_{\mathbf{k},p}a_{\mathbf{k},p}|0\rangle + \frac{1}{2}\langle 0|\mathbb{1}|0\rangle\right)\end{aligned} \tag{4.16}$$

The first term clearly vanishes, since the annihilation operator acts on the vacuum to produce 0. The trouble arises with the second term, which gives a constant $1/2$. The sum is over an infinite number of modes, so this blows up and tells us that the vacuum energy is infinite! As strange as this sounds, this is genuinely the result that we obtain from quantum electrodynamics. Fortunately, decades of work in this field have provided us with three pretty handy ways to deal with this.

Just Throw It Away I wish I was kidding, but this is genuinely one way to deal with the infinite term, and the reasoning behind it is pretty sound. The infinite vacuum energy only showed up as a result of us trying to find the *absolute* ground state energy. However, measurements in reality are always only defined *relative* to some known reference point. That is, to say that a certain excited state has energy E is really the statement that that state has E more energy with respect to the ground state. If we simply define this infinite vacuum ground energy as our reference point, then the infinity never shows up again in any other energies and the problem is solved. This infinite term is sometimes also playfully called the **cosmological constant**.

Normal Ordering We start by rewriting the Hamiltonian of Eq. 4.12 as:

$$\begin{aligned}H_{\mathbf{k},p} &= \frac{1}{2}\hbar\omega(\mathbf{k})\left(a^\dagger_{\mathbf{k},p}a_{\mathbf{k},p} + a^\dagger_{\mathbf{k},p}a_{\mathbf{k},p} + 1\right) \\ &= \frac{1}{2}\hbar\omega(\mathbf{k})\left(a^\dagger_{\mathbf{k},p}a_{\mathbf{k},p} + a_{\mathbf{k},p}a^\dagger_{\mathbf{k},p}\right)\end{aligned}$$

We now define the **normal-ordered** version $:A:$ of an operator A to be that in which all constituent annihilation operators are placed to the right of any creation operators. Writing the Hamiltonian in normal order then produces:

$$:H_{\mathbf{k},p}: = \hbar\omega(\mathbf{k})a^\dagger_{\mathbf{k},p}a_{\mathbf{k},p} \tag{4.17}$$

In doing so, we have gotten rid of the infinite term once again, but at the expense of some abstraction with the definition of normal-ordering. It turns out this is usually the method of choice in quantum field theory, since normal ordering is ideal for deriving Feynman rules and computing Feynman diagrams.

Renormalisation I won't go into the details of this because it typically requires an entire semester's course to properly explain[15]. The rough idea is that once a full quantum treatment is accorded to the Hamiltonian (quantisation of the Coulomb interaction was ignored here, so this is merely semiclassical), alongside a relativistic quantum treatment for the interacting fermions (such as the electrons which form the elementary charges in our system), we obtain a fully quantised theory of quantum electrodynamics. In such a theory, there are processes whereby photons can polarise the vacuum state, producing particles and antiparticles which exist briefly before annihilating each other to produce a photon again. This infinite term appears in such a process, whereby the propagation of a single photon needs to be thought of as the sum of infinitely many processes of pair production and annihilation, resulting in an infinite electric charge for the electron. Julian Schwinger, Shin'ichirō Tomonaga and Richard Feynman independently proved that these infinities could be effectively dealt with by introducing new techniques of summing over these processes, and were jointly awarded the 1965 Nobel Prize in Physics for their work.

While renormalisation is the rigorous choice, we are working toward a quantum optics picture of electrodynamics, so that level of rigour is not required and it is perfectly justified to simply throw the diverging constant away.

5 States of the Electromagnetic Field

I promise you, the hardest part is now behind us. While the whole process of quantising the EM field was great for obtaining a corpuscular theory of light, in order for it to be truly universal, we need to be able to recover classical behaviours from this quantised theory.

5.1 Semiclassical Coherent States

For this, consider the 'simplest' case of classical light – a beam of monochromatic light emitted by a laser. This has a well-defined phase and amplitude, but it does not seem to easily fit into our notion of the number states of the EM field. Thus, we want to find the quantum state which is *most classical* (minimally deviating from the classical expectation value). This quantum state $|\alpha\rangle$ should then obey the correspondence principle, when applied to the Hamilton's equations from Eq. 4.4:

$$\begin{cases} \langle \dot{q} \rangle_\alpha &= \langle p \rangle_\alpha \\ \langle \dot{p} \rangle_\alpha &= -\omega^2 \langle q \rangle_\alpha \end{cases} \quad \text{where} \quad \langle q \rangle_\alpha = \langle \alpha | q | \alpha \rangle \quad \langle p \rangle_\alpha = \langle \alpha | p | \alpha \rangle \tag{5.1}$$

The simplest first guess is to try a number state, so we first rewrite the position and momentum operators in terms of the creation and annihilation operators (dropping all subscripts $(\mathbf{k}, p)$ since it's clear we only care about a single mode and there is no coupling to other modes):

$$\begin{cases} q &= \sqrt{\frac{\hbar}{2\omega}}(a + a^\dagger) \\ p &= -i\sqrt{\frac{\hbar\omega}{2}}(a - a^\dagger) \end{cases} \implies \begin{cases} \langle q \rangle_n &= \langle n | \sqrt{\frac{\hbar}{2\omega}}(a + a^\dagger) | n \rangle = 0 \\ \langle p \rangle_n &= \langle n | (-i)\sqrt{\frac{\hbar\omega}{2}}(a - a^\dagger) | n \rangle = 0 \end{cases} \quad \text{using} \quad \langle n' | n \rangle = \delta_{nn'} \tag{5.2}$$

There is no violation of Hamilton's equations of motion here, since both sides are trivially zero. However, these are clearly not suitable for describing classical states since they all correspond to the origin of phase space! Classically, we know that an oscillator traverses phase space along an ellipse centred at the origin, in a counterclockwise direction. Thus, all number states effectively correspond to a "zero orbit", which is not what we expect from a classical state. Interestingly, this does point us at the fact that any classical trajectory (such as

[15] See PHY 510.

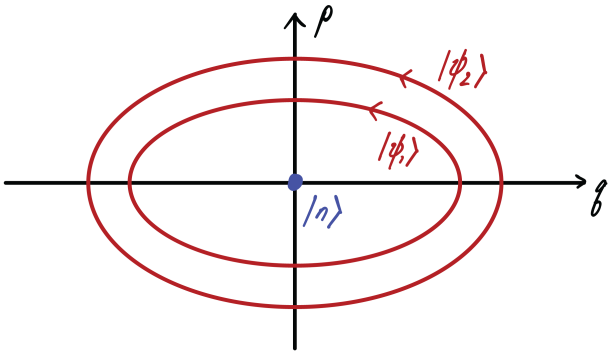

FIGURE 1: Orbits in Optical Phase Space. The number states lie on a trivial trajectory which sit at the origin, so they clearly cannot correspond to the semiclassical orbits which resemble that of a classical harmonic oscillator. We wish to find states such as $|\psi_1\rangle$ and $|psi_2\rangle$ whose orbits look classical, while admitting a quantum state description within the quantised formalism.

the one which remains at the origin) can correspond to many quantum states, which is somewhat reminiscent of statistical mechanics. An example of this is shown in Fig. 1. With this in mind, we can ask the following question – Given a specific orbit, what is the state of the quantum harmonic oscillator which exhibits phase space dynamics that most resembles *that* orbit? To answer this, we consider the energy of any state on the orbit:

$$\begin{aligned}
\langle H\rangle &= \frac{1}{2}\left\langle p^2+\omega^2 q^2\right\rangle \\
&= \frac{1}{2}\left\langle p^2\right\rangle + \frac{1}{2}\omega^2\left\langle q^2\right\rangle \\
&\geqslant \frac{1}{2}\left\langle p\right\rangle^2 + \frac{1}{2}\omega^2\left\langle q\right\rangle^2 \quad \text{where the equality only holds for a classical system} \\
&= E_{\text{classical}}
\end{aligned}$$

Thus, the job of finding the most classical state $|\alpha\rangle$ is akin to minimising $\langle H\rangle$ on a given phase space orbit. With our quantised Hamiltonian for the EM field, we are trying to minimise:

$$\langle H\rangle_\alpha = \hbar\omega\,\langle\alpha|a^\dagger a|\alpha\rangle \tag{5.3}$$

We start by sandwiching an identity between $a^\dagger$ and a so that we can invoke the completeness relation. However, in using the completeness relation, we are entirely free to choose what basis we want to expand the identity in. Choose the basis $\{|\psi_n\rangle\}$ where $|\psi_1\rangle = |\alpha\rangle$, and all other states are orthogonal to this (i.e. $\langle\psi_j|\psi_1\rangle = \delta_{1,j}$). Then, we proceed as:

$$\begin{aligned}
\langle H\rangle &= \sum_{j=1}^{\infty}\hbar\omega\,\langle\alpha|a^\dagger|\psi_j\rangle\,\langle\psi_j|a|\alpha\rangle \\
&= \hbar\omega|\,\langle\alpha|a|\alpha\rangle\,|^2 + \hbar\omega\sum_{j>1}^{\infty}|\,\langle\psi_j|a|\alpha\rangle\,|^2
\end{aligned}$$

Note that the second term is always non-negative, and in trying to minimise $\langle H\rangle$, the best we can do is to have the second term vanish. There are two cases to consider now:

1. If $|\alpha\rangle$ is an eigenstate of a, then: $|\,\langle\psi_j|a|\alpha\rangle\,|^2 = |\,\langle\psi_j|\alpha\rangle\,|^2|\alpha|^2 = 0$

2. If $|\alpha\rangle$ is not an eigenstate of a, then: $|\langle\psi_j|a|\alpha\rangle|^2 = |\langle\psi_j|a|a^{\parallel}\rangle + \langle\psi_j|a|a^{\perp}\rangle| > 0$

where $|a^{\parallel}\rangle$ and $|a^{\perp}\rangle$ are the projection and orthogonal components respectively of the action of a on $|\alpha\rangle$. Thus, the state that minimises the energy expectation value is an eigenstate of the annihilation operator! This leads us to the defining property of the semiclassical states, also known as **coherent states** – a coherent state $|\alpha\rangle$ is any state that is an eigenstate of the annihilation operator $\hat{a}$ with eigenvalue α. Mathematically, we write:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{5.4}$$

5.2 Properties of Coherent States

Coherent states might be eigenstates of $\hat{a}$, but since $\hat{a}$ is non-Hermitian, the properties of the states themselves are not entirely trivial. Firstly, the non-hermiticity of $\hat{a}$ allows its eigenvalues to be complex, namely we are allowed to have $\alpha \in \mathbb{C}$. To make matters worse, the set of eigenstates of $\hat{a}$ is not guaranteed to be complete. Thus, it's wise to obtain $|\alpha\rangle$ in the representation of a well-known complete basis – the number states. We start from Eq. 5.4:

$$\begin{aligned}
a|\alpha\rangle &= \alpha|\alpha\rangle \\
\sum_n a c_n |n\rangle &= \sum_n \alpha c_n |n\rangle \quad \text{assuming} \quad \alpha = \sum_{n=0}^{\infty} c_n |n\rangle, c_n \in \mathbb{C} \\
\sum_n c_n \sqrt{n}|n-1\rangle &= \sum_n \alpha c_n |n\rangle \quad \text{using the definition of the lowering operator}
\end{aligned}$$

Ignoring the $n = 0$ term (which is destroyed on the left), we obtain the recursion relation:

$$\begin{aligned}
c_{n+1} &= \frac{\alpha}{\sqrt{n+1}} c_n \\
\implies c_n &= \frac{\alpha^n}{\sqrt{n!}} c_0
\end{aligned}$$

where we can generate all coefficients via the recursion relation. We can now set c_0 by the overall normalisation of the state:

$$\begin{aligned}
1 = \langle\alpha|\alpha\rangle &= \sum_n \frac{\left(|\alpha|^2\right)^n}{n!} |c_0|^2 = e^{|\alpha|^2} |c_0|^2 \\
\implies c_0 &= e^{-|\alpha|^2/2}, \quad \text{ignoring any complex phase of } c_0
\end{aligned}$$

Finally, this produces the decomposition of the coherent state into the number states as:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{5.5}$$

With the definition of the coherent state and some basic properties, we can address the earlier issue of *The Curious Case of the Vanishing Expectation Value*. These are now easily computed:

$$\begin{cases}
\langle q\rangle_\alpha &= \langle\alpha|\sqrt{\frac{\hbar}{2\omega}}(a + a^\dagger)|\alpha\rangle = \sqrt{\frac{\hbar}{2\omega}}(\alpha + \alpha^*) = \sqrt{\frac{\hbar}{\omega}}\,\mathrm{Re}\,\alpha \\
\langle p\rangle_\alpha &= \langle\alpha|(-i)\sqrt{\frac{\hbar\omega}{2}}(a - a^\dagger)|\alpha\rangle = -i\sqrt{\frac{\hbar\omega}{2}}(\alpha - \alpha^*) = \sqrt{2\hbar\omega}\,\mathrm{Im}\,\alpha
\end{cases} \tag{5.6}$$

Now, we know the coherent states cannot possibly be eigenstates of the Hamiltonian (since it is an eigenstate of the annihilation operator, which does not commute with H), so they have to evolve over time. We can probe their evolution by applying the propagator:

$$\begin{aligned} e^{-iHt/\hbar} &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-iHt/\hbar} |n\rangle \\ &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\hbar\omega\left(n+\frac{1}{2}\right)t/\hbar} |n\rangle \\ &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega nt} e^{-i\omega t/2} |n\rangle \\ &= e^{-|\alpha|^2/2} e^{-i\omega t/2} \sum_n \frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle \\ &= e^{-i\omega t/2} \left|\alpha e^{-i\omega t}\right\rangle \end{aligned} \tag{5.7}$$

Thus, a coherent state evolves under the free Hamiltonian to become another coherent state (up to a global phase). The expectation values thus evolve over time as:

$$\begin{cases} \langle q\rangle_\alpha (t) &= \cos(\omega t) \langle q\rangle_\alpha (0) + \frac{1}{\omega} \sin(\omega t) \langle p\rangle_\alpha (0) \\ \langle p\rangle_\alpha (t) &= \cos(\omega t) \langle p\rangle_\alpha (0) - \omega \sin(\omega t) \langle q\rangle_\alpha (0) \end{cases} \tag{5.8}$$

The dynamics here describe the trajectory of an elliptical path in phase space, which is the solution to the initial problem we had! Indeed, the coherent states trace out classical orbits in phase space, so they are the semiclassical states we were looking for. Interestingly, while these states *technically* don't form a complete basis in the sense that we're used to, they form an **overcomplete basis**. To prove this, we need to show the following:

1. The coherent states form a complete basis:

$$\frac{1}{\pi} \int \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| = \mathbb{1} \quad \text{where } \mathrm{d}^2\alpha \equiv \mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha) \tag{5.9}$$

2. No two coherent states are orthogonal:

$$\langle\alpha|\beta\rangle \neq 0 \text{ for } \alpha \neq \beta \tag{5.10}$$

The second is easier to prove, so let's start with that. This is easily checked using the expansion from Eq. 5.5 to write both states in terms of the number states:

$$\begin{aligned} \langle\alpha|\beta\rangle &= \sum_{n,m} e^{-|\alpha|^2/2} e^{-|\beta|^2/2} \frac{(\alpha^*)^n (\beta)^m}{\sqrt{n!m!}} \langle n|m\rangle \\ &= \sum_n e^{-\left(|\alpha|^2+|\beta|^2\right)/2} \frac{(\alpha^*\beta)^n}{n!} \\ &= e^{-\left(|\alpha|^2+|\beta|^2\right)/2} e^{\alpha^*\beta} \neq 0, \quad \text{for all} \quad \alpha, \beta \in \mathbb{C} \end{aligned} \tag{5.11}$$

Thus, any two coherent states will always have some overlap, though this can be exponentially small. The proof of completeness is also rather straightforward, though we have to be a little creative with the math. We start as:

$$\begin{aligned}
\int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| &= \sum_{n,m} \int \mathrm{d}^2\alpha\, \frac{1}{\sqrt{n!m!}} \alpha^n \alpha^{*m}\, |n\rangle\langle m|\, e^{-|\alpha|^2} \\
&= \sum_{n,m} \frac{1}{\sqrt{n!m!}} \int r\mathrm{d}r\mathrm{d}\phi\, r^{n+m} e^{i\phi(n-m)} e^{-r^2}\, |n\rangle\langle m| \quad \text{where we let } \alpha = re^{i\phi} \\
&= \sum_{n,m} \frac{1}{\sqrt{n!m!}} \underbrace{\int_0^\infty \mathrm{d}r\, r^{n+m+1} e^{-r^2}}_{\text{radial}} \underbrace{\int_0^{2\pi} \mathrm{d}\phi\, e^{i\phi(n-m)}\, |n\rangle\langle m|}_{\text{angular}}
\end{aligned} \tag{5.12}$$

It is wiser to deal with the angular integral first, since this will help us with the radial integral later. The angular integral is more straightforward:

$$\begin{aligned}
\int_0^{2\pi} \mathrm{d}\phi\, e^{i\phi(n-m)} &= \frac{1}{i(n-m)}\, e^{i\phi(n-m)}\Big|_{\phi=0}^{\phi=2\pi} \\
&= \frac{1}{i(n-m)} \left[e^{2\pi i(n-m)} - 1\right] \\
&= \frac{2e^{i\pi(n-m)}}{n-m} \sin\left[\pi(n-m)\right] \\
&= 2\pi e^{i\pi(n-m)} \mathrm{sinc}[\pi(n-m)] \\
&= 2\pi\delta_{mn}
\end{aligned} \tag{5.13}$$

where in the last line, we use the fact that $(n-m)$ is an integer, and the zeroes of $\mathrm{sinc}(\pi x)$ occur at all integer values of x except 0. Using this result, the radial integral is rather easy to perform:

$$\begin{aligned}
\int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| &= \sum_n \frac{2\pi}{n!} \int_0^\infty \mathrm{d}r\, r^{2n+1} e^{-r^2}\, |n\rangle\langle n| \\
&= \sum_n \frac{\pi}{n!} \int_0^\infty \mathrm{d}y\, y^{n+\frac{1}{2}} e^{-y} \frac{1}{\sqrt{y}}\, |n\rangle\langle n| \quad \text{where we let } y = r^2 \\
&= \sum_n \frac{\pi}{n!} \int_0^\infty \mathrm{d}y\, y^n e^{-y}\, |n\rangle\langle n| \\
&= \sum_n \frac{\pi}{n!} \Gamma(n+1)\, |n\rangle\langle n| \quad \text{using the integral definition of the } \Gamma \text{ function} \\
&= \pi \mathbb{1}
\end{aligned} \tag{5.14}$$

which produces the required result. Thus, we have shown that the coherent states form an overcomplete basis, in that they are complete yet nonorthogonal. Having established that they indeed cover the Hilbert space within a single mode of the EM field, we can try to establish a more intuitive link between the coherent states and the number states. Specifically, we can ask – how many photons would one expect to observe in a coherent state?

For a given coherent state, we can easily evaluate this as:

$$\langle n \rangle_\alpha = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2 \tag{5.15}$$

This result can also be obtained in another fashion, which provides more insight into why this is the case. We had the decomposition of a coherent in terms of the number states given by Eq. 5.5. From this, we can write the probability of observing a photon number n by taking the absolute square of the coefficients as:

$$p_n = e^{-|\alpha|^2} \frac{\left(|\alpha|^2\right)^n}{n!} \tag{5.16}$$

$$= e^{-\mu} \frac{\mu^n}{n!} \quad \text{where we let} \quad \alpha = \sqrt{\mu} e^{i\varphi} \tag{5.17}$$

which is just a Poisson distribution! This is well characterised as having a mean and variance equal to μ, and this agrees with the result from Eq. 5.15 above. For this reason, light from a coherent state is also called **Poissonian light**. Similarly, a number state (Fock state) of fixed number n will have zero variance and consequently cannot be described in a classical fashion, so this is sometimes called **sub-Poissonian light**.

5.3 Thermal States

For the sake of completeness, I have to include this bit, though you can feel free to skip it if you haven't covered statistical mechanics yet. We have talked about the *most quantum* state (Fock states) and the *most classical* state (coherent states). What is the state of the EM field that is simply *most commonly found*? We know that even within a perfect cavity devoid of light sources, radiation is found to be emitted by the cavity itself at any nonzero temperature – blackbody radiation. The quantum state corresponding to blackbody radiation is known as a **thermal state**.

Since a thermal state represents blackbody radiation, it must follow the distribution of energy predicted by statistical mechanics. This is easily represented by the density matrix:

$$\rho_T = \sum_n p_n(T) \, |n\rangle\langle n| \tag{5.18}$$

where $p_n(T)$ is a temperature-dependent probability distribution. For a system in contact with a heat bath at temperature T, this distribution is none other than that obtained from the canonical partition function:

$$p_k(T) = \frac{1}{Z} e^{-\beta E_k} \quad \text{where} \quad \beta = \frac{1}{k_B T}$$

$$= \frac{e^{-\beta E_k}}{\mathrm{Tr}\left(e^{-\beta H}\right)} \tag{5.19}$$

Here, the canonical probability depends on the Hamiltonian, which itself is a sum of contributions from all modes. We can use the Baker-Campbell-Hausdorff formula now to easily separate this as:

$$H = \sum_{\mathbf{k},p} \hbar\omega(\mathbf{k}) a^\dagger_{\mathbf{k},p} a_{\mathbf{k},p} = \sum_{\mathbf{k},p} \hbar\omega(\mathbf{k}) n_{\mathbf{k},p}$$

$$\implies e^{-\beta H} = e^{-\beta \sum_{\mathbf{k},p} \hbar\omega(\mathbf{k}) n_{\mathbf{k},p}}$$

$$= \bigotimes_{\mathbf{k},p} e^{-\beta \hbar\omega(\mathbf{k}) n_{\mathbf{k},p}}$$

$$= \bigotimes_{\mathbf{k},p} e^{-\beta H_{\mathbf{k},p}} \tag{5.20}$$

Thus, instead of having to consider the global state ρ_T and the full partition function from $e^{-\beta H}$, we can focus on a single subspace corresponding to a single mode, writing $\rho_{\mathbf{k},p,T}$ and the segment of the partition function from $e^{-\beta H_{\mathbf{k},p}}$. With this in mind, we can write the thermal state as (dropping all subscripts since we are looking at a single mode):

$$\begin{aligned}\rho &= \sum_n \frac{e^{-\beta\hbar\omega n}}{\mathrm{Tr}\left(e^{-\beta\hbar\omega n}\right)} |n\rangle\langle n| \\ &= \sum_n \frac{e^{-\beta\hbar\omega n}}{\sum_j e^{-\beta\hbar\omega j}} |n\rangle\langle n| \\ &= \left(1 - e^{-\beta\hbar\omega}\right) \sum_n e^{-\beta\hbar\omega n} |n\rangle\langle n| \end{aligned} \tag{5.21}$$

This gives us the density matrix for a single mode at some fixed temperature T, in terms of the inverse temperature $\beta = 1/k_B T$. Since the density matrix here is still a diagonal sum over the number states (purely classical probability distribution), we can easily use this to write the expectation value of the photon number:

$$\begin{aligned}\langle n\rangle_T &= \sum_n p_n n \\ &= \sum_n \left(1 - e^{-\beta\hbar\omega}\right) e^{-\beta\hbar\omega n} n \\ &= \sum_n n\alpha^n - n\alpha^{n+1} \quad \text{where} \quad \alpha = e^{-\beta\hbar\omega} \\ &= \frac{\alpha}{(1-\alpha)^2} - \frac{\alpha^2}{(1-\alpha)^2} \\ &= \frac{\alpha}{1-\alpha} \\ &= \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \\ &= \frac{1}{e^{\beta\hbar\omega} - 1} \end{aligned} \tag{5.22}$$

which is the exact number distribution expected from Bose-Einstein statistics! To be fair, there was nothing intrinsically quantum here since the density matrix was always diagonal, it was merely a tool to aid the formulation of the problem. Having understood the primary properties of the various types of light, there is perhaps just one last question that is worth tackling in this introductory review. Suppose we have a coherent state $|\alpha\rangle$ with $\alpha = 0$, also known as the coherent vacuum. How is this different from the vacuum of Fock space $|0\rangle$, if at all? Notably, we have not yet dealt with ensuring that these states obey the uncertainty principle in the generalised coordinates and momenta (q, p). In order to do this, we need a (very) brief review on some aspects of the Hamiltonian formalism.

5.4 Interlude: Action-Angle Variables

In the Hamiltonian formalism, the Hamiltonian is expressed in terms of a set of generalised coordinates $\{q_i\}$ and associated generalised momenta $\{p_i\}$ that satisfy Hamilton's equations of motion from Eq. 2.2. However, there is nothing really special about the coordinates $\{q_i, p_i\}$, and we could just as well use another set of coordinates $\{Q_i, P_i\}$ by a suitable transformation. Such coordinate transformations will not (in general) preserve the form of the equations of motion, but any transformation that does is known as a **canonical transformation**. Specifically, the transformation may modify the form of the Hamiltonian as:

$$H(q_i, p_i, t) \longrightarrow K(Q_i, P_i, t) \Longrightarrow \begin{cases} \dot{Q}_i &= \frac{\partial K}{\partial P_i} \\ \dot{P}_i &= -\frac{\partial K}{\partial Q_i} \end{cases} \tag{5.23}$$

where $K(Q_i, P_i, t)$ is known as the **Kamiltonian**[16] [6]. The natural question at this point is – why bother going through these transformations if it's just math? Well, it turns out that some canonical coordinates are less annoying than others. Suppose you found a particular coordinate pair (J_i, θ_i) such that the Kamiltonian becomes a function of the set of $\{J_i\}$ alone. Then, the equations of motion reduce to:

$$\begin{cases} \dot{J}_i = \frac{\partial K}{\partial \theta_i} = 0 & \implies J_i = \text{const.} \\ \dot{\theta}_i = \frac{\partial K}{\partial J_i} = \omega(J_i) = \text{const.} & \implies \theta_i = \omega(J_i)t + \text{const.} \end{cases} \tag{5.24}$$

In this form, all $\{J_i\}$ are constants, while the evolution of all $\{\theta_i\}$ are linear in time! We call these action variables and angle variables respectively. This action-angle canonical transformation is particularly useful when a system undergoes some fixed periodic motion in phase space (libration) and we want to just determine the periodicity of this motion. It can be shown (using the Hamilton-Jacobi equation, which I will not go into) that by defining:

$$J_i = \oint \mathrm{d}q_i \, p_i \tag{5.25}$$

with the closed loop signifying one complete libration in phase space, so the action variable is simply related to the area enclosed by the loop in phase space. The angle variable θ_i then plays the role of a linear parametrisation as the system traverses the loop, with the change in θ_i over one full libration being:

$$\Delta\theta_{i,\text{cycle}} = 1 \implies \omega\tau = 1 \quad \text{where } \tau \text{ is the period} \tag{5.26}$$

Thus, the function $\omega(J_i)$ is the frequency of the periodic motion! In our original problem, we saw that a coherent state evolves over time with constant energy, staying on an ellipse of fixed energy in phase space. Mathematically, we can say:

$$\begin{aligned} H(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2) &= \left(\frac{p}{\sqrt{2}}\right)^2 + \left(\frac{\omega q}{\sqrt{2}}\right)^2 = E \\ \implies \underbrace{\left(\frac{p}{\sqrt{p^2+\omega^2q^2}}\right)^2}_{\sin^2\varphi} &+ \underbrace{\left(\frac{\omega q}{\sqrt{p^2+\omega^2q^2}}\right)^2}_{\cos^2\varphi} = 1 \end{aligned} \tag{5.27}$$

where we have nicely circularised the phase space orbit, and we obtain φ as the angle variable describing the dynamics of (q, p) as required! It can be shown through some tedious differentiation that this circular phase space orbit obeys the action-angle variable equations of motion from Eq. 5.24.

5.5 Quadratures

It turns out that the description of coherent states is more neatly performed in the formalism of action-angle variables, since we know that these states traverse classical orbits in phase space. In the previous sections, we used a (q, p) coordinate pair with different dimensions. We can reduce this coordinate pair to dimensionless variables by defining new variables:

$$Q = \sqrt{\frac{2\omega}{\hbar}}q, \quad P = \sqrt{\frac{2}{\hbar\omega}}p \tag{5.28}$$

With these variables, the Hamiltonian becomes:

[16] This genuinely began as a joke by Herbert Goldstein in his textbook, but the name has somehow stuck.

$$H(Q,P) = \frac{\hbar\omega}{4}\left(P^2+Q^2\right) \implies \left(\frac{P}{\sqrt{P^2+Q^2}}\right)^2 + \left(\frac{Q}{\sqrt{P^2+Q^2}}\right)^2 = 1 \tag{5.29}$$

which preserves the canonical form while placing Q and P on equal footing. In these coordinates, the creation and annihilation operators become:

$$\begin{aligned} a &= \frac{1}{2}(Q+iP), \quad a^\dagger = \frac{1}{2}(Q-iP) \\ Q &= a + a^\dagger, \quad P = -i(a-a^\dagger) \\ \implies [Q,P] &= 2i\mathbb{1} \end{aligned} \tag{5.30}$$

From the generalised uncertainty principle, we can then set a condition on the variances of these new coordinates as:

$$\sigma_Q \sigma_P \geqslant \left|\frac{1}{2i}\langle[Q,P]\rangle\right| = 1 \tag{5.31}$$

It is not difficult to show that a regular coherent state saturates this inequality (that is, it satisfies the equality case). Now comes the interesting bit – Q and P are both Hermitian, and thus observables (since they are just rescaled versions of q and p). With the angle variable parametrisation from Eq. 5.27, we can identify Q as the case where $\varphi = 0$, and P as the case where $\varphi = \pi/2$. In this form, Q and P are known as the **quadratures** of optical phase space. Since any coherent state can be expressed as $|\alpha\rangle, \alpha \in \mathbb{C}$, we could equivalently write:

$$|\alpha\rangle = \left|\sqrt{\mu}e^{i\phi}\right\rangle, \mu \in \mathbb{R}, \phi \in [0, 2\pi) \tag{5.32}$$

This would produce expectation values for the quadratures as:

$$\langle Q\rangle_\alpha = 2\sqrt{\mu}\cos\phi, \quad \langle P\rangle_\alpha = 2\sqrt{\mu}\sin\phi \tag{5.33}$$

More importantly, the fluctuations in the quadratures are given by:

$$(\Delta Q)^2 = \left\langle Q^2\right\rangle_\alpha - \langle Q\rangle_\alpha^2 = 1 \implies (\Delta Q)_\alpha = 1 \tag{5.34}$$

with a similar expression for the fluctuation in P. This agrees with Eq. 5.31, where I stated that the coherent states saturate the uncertainty relation.

At this point, all the math is done and we can finally talk about the physical meaning of what we just did. In defining the action-angle variables and quadratures, we converted our Hamiltonian into a dimensionless form that was parametrised by two variables, only one of which explicitly described the motion of the state in phase space – the angle variable. The angle variable defines a rotation angle of the state as it traverses the circularised phase space orbit, and is linear in time. We defined the quadratures from this formalism to more easily determine the uncertainty in the state along the canonical variable axes (q,p), which have become (Q,P) in the dimensionless formalism. While the expectation values of the two quadratures (just the usual coordinate axes in our case) follow the elliptical path, the uncertainty maintains a constant value of 1, which implies that the coherent state is not a single point traversing the ellipse, but rather some fuzzy disc! This is shown in Fig. 2. As the disc traverses the circular path, we can characterise it by making measurements along either quadrature, but there will always be some degree of uncertainty to which we can measure it.

For a coherent state to be "very classical", we expect there to be many photons in the state such that it no longer resembles a Fock state. From Eq. 5.15, this implies that the value of $|\alpha| = \sqrt{\mu}$ is extremely large.

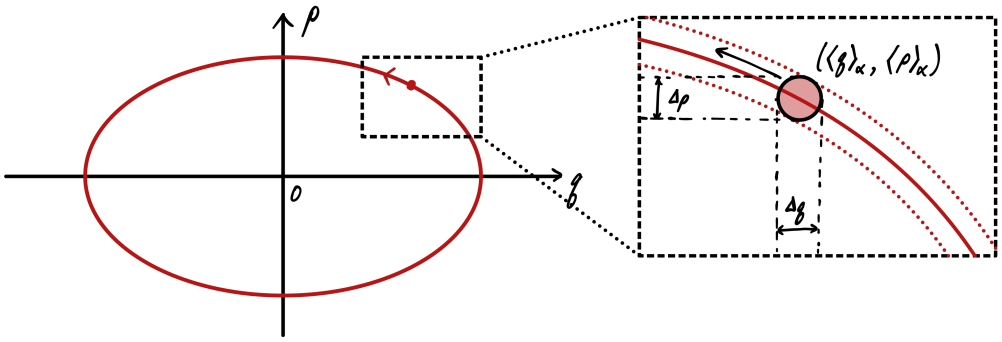

FIGURE 2: Coherent States in Optical Phase Space. A coherent state traverses the optical phase space on an ellipse in a constant fashion. It maintains a fixed level of uncertainty, regardless of the ellipse's size, so a state with large $\sqrt{\mu}$ resembles a classical state.

Intuitively, this should resemble a classical orbit, since the "fuzziness" of the disc maintains a constant size regardless of how large the orbit is. Thus, as the number of photons in the coherent state grows extremely large, the uncertainty in the state becomes negligible compared to the expectation values and the dynamics look classical! On the other hand, if we consider the extreme case of the coherent vacuum, the orbit is trivially at the origin. However, the fluctuations prevent us from observing purely zero along either quadrature, and so we become maximally sensitive to the fluctuations in the coherent state. This is a great demonstration of the fact that vacuum is not empty, but instead contains fluctuations in the underlying quantum fields, such as the electromagnetic field in this case, and is only explicable in a quantum mechanical framework.

While the uncertainty principle in Eq. 5.31 sets an absolute lower bound on the total product of the uncertainties, it is possible to lower the uncertainty in one quadrature at the expense of blowing it up along the other. States which exhibit this are known as **squeezed coherent states** and are notoriously hard to generate in a laboratory setting, but advances in atomic and optical physics have made it tractable. For instance, the coherent vacuum was demonstrably squeezed as early as 1997 [7], proving the non-emptiness of the quantum vacuum. On a final note, we can answer the question of how the number states (Fock states) which occupy the trivial orbit at the origin of phase space satisfy the uncertainty principle. It turns out that the number states are not adequately described in the quadrature formalism, and instead obey a number-phase uncertainty relation:

$$\Delta N \Delta\theta \geqslant 1 \tag{5.35}$$

Near-perfectly generated Fock states turn out to be highly number-squeezed states, which in turn forces their phase uncertainty to blow up, smearing them over an entire orbit if one were to try and visualise this in phase space. The definition of the phase operator, however, is still a subject of intense debate in the quantum optics community, so one needs to be careful with this interpretation.

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